Debre Berhan University
Applied Mathematics One
Lecture note: Chapter 3

"Mathematics is the most beautiful and most powerful creation of the human spirit."

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March, 2020

Contents

Chapter 1

Limit and Continuity

1.1 Definition of Limit

As the precise definition of a limit is a bit technical, it is easier to start with an informal definition; we'll explain the formal definition later.

We suppose that a function f is defined for x near c (but we do not require that it be defined when $x = c$).

Definition 1.1: (Informal definition of a limit)

We call L the limit of $f(x)$ as x approaches c if $f(x)$ becomes close to L when x is close (but not equal) to c , and if there is no other value L' with the same property.
When this holds we write

$$\lim_{x\to c} f(x) = L$$

or

$$f(x) \to L \quad \text{as} \quad x \to c$$

Notice that the definition of a limit is not concerned with the value of $f(x)$ when $x = c$ (which may exist or may not). All we care about are the values of $f(x)$ when x is close to c , on either the left or the right (i.e. less or greater).

Definition 1.2

Let f be a function defined on some open interval that contains the number c, except possibly at c itself. Then we say that the limit of f as x approaches c is L, and we write

$$\lim_{x\to c} f(x) = L$$

if for every number $\epsilon > 0$ there is a number $\delta > 0$ such that

$$\text{if } 0 <| x - c |< \delta \text{ then } | f(x) - L |< \epsilon$$

Since $| x - c |$ is the distance from x to c and $| f(x) - L |$ is the distance from $f(x)$ to L,and ϵ can be arbitrarily small, the definition of a limit can be expressed in words as follows:
$\lim_{x\to c} f(x) = L$ means that the distance between $f(x)$ and L can be made arbitrarily small by taking the distance from x to a sufficiently small (but not 0's).
Alternatively, $\lim_{x\to c} f(x) = L$ means that the values of $f(x)$ can be made as close as we please to L by taking x close enough to c (but not equal to c). We can also reformulate definition in terms of intervals by observing that the inequality $| x - c |< \delta$ is equivalent to $-\delta < x - c < \delta$

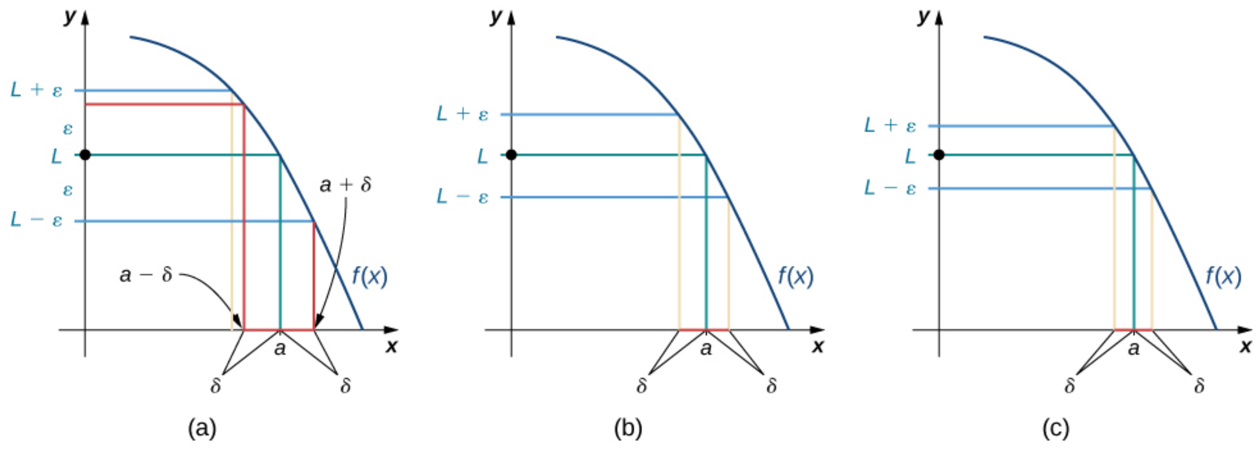


Figure 1.1: The $\epsilon - \delta$ definition of the limit of $f(x)$ as x approaches a

, which in turn can be written as $c - \delta < x < c + \delta$. Also $0 <| x - c |$ is true if and only if $x - c \neq 0$, that is, $x \neq a$. Similarly, the inequality $| f(x) - L |< \epsilon$ is equivalent to the pair of inequalities $L - \epsilon < f(x) < L + \epsilon$. Therefore, in terms of intervals, definition can be stated as follows:
$\lim_{x \to c} f(x)$ means that for every $\epsilon > 0$ (no matter how ϵ is small) we can find $\delta > 0$ such that if x lies in the open interval $(c-\delta, c+\delta)$ and $x \neq c$, then $f(x)$ lies in the open interval $(L-\epsilon, L+\epsilon)$

Example 1.1

Prove that $\lim_{x\to 3}(4x-5) = 7$.
Solution:-
Let ϵ *be a given positive number. We want to find a number* δ *such that*
$|(4x-5)-7| < \epsilon$ *whenever* $0 < |x-3| < \delta$. *But* $|(4x-5)-7| = |4x-12| = 4|x-3|$.
Therefore, we want $4|x-3| < \epsilon$ *whenever* $|x-3| < \delta$ *this implies* $|x-3| < \dfrac{\epsilon}{4}$ *whenever* $|x-3| < \delta$ *This suggests that we should choose* $\delta = \dfrac{\epsilon}{4}$.

Example 1.2

Using formal definition of limit prove that $\lim_{x\to 2} x^2 = 4$

Solution:

We must show that for any $\epsilon > 0$ *there exists* $\delta > 0$ *such that* $|x^2 - 4| < \epsilon$ *when ever* $0 < |x-2| < \delta$ *Factoring, we get* $|x^2 - 4| = |x-2||x+2|$ *we want to show that* $|x^2-4|$ *is small when* x *is close to 2. To do this, we first find an upper bound for the factor* $|x+2|$. *If* x *is close to 2, we know that the factor* $|x-2|$ *is small, and that the factor* $|x+2|$ *is close to 4. Because we are considering values of* x *close to 2s, we can concern ourselves with only those values of* x *for which* $|x-2| < 1$; *that is, we are requiring the* δ, *for which we are looking, to be less than or equal to 1. The inequality* $|x-2| < 1$ *is equivalent to* $-1 < x-2 < 1$ *which is equivalent to* $1 < x < 3$ *or, equivalently,* $3 < x+2 < 5$. *This means that if* $|x-2| < 1$, *then* $3 < |x+2| < 5$ *therefore, we have* $|x^2-4| = |x-2||x+2| < 5|x-2|$. *Now we want* $5|x-2| < \epsilon$ *or, equivalently* $|x-2| < \frac{1}{5}\epsilon$. *Thus, if we choose* δ *to be the smaller of 1 and* $\frac{\epsilon}{5}$, *then when ever* $|x-2| < \delta$, *it follows that* $|x-2| < \frac{1}{5}\epsilon$ *and* $|x+2| < 5$ *because this is true when* $|x-2| < 1$ *and so* $|x^2-4| < \frac{\epsilon}{5}$. *Therefore, we conclude that* $|x^2-4| < \epsilon$ *whenever* $0 < |x-2| < \delta$ *if* δ *is the smaller of the two numbers 1 and* $\frac{\epsilon}{5}$, *which we write as* $\delta = min(1, \frac{\epsilon}{5})$.

Exercise 1.1

Prove that

a) $\lim_{x\to 3}(-2x+7) = 1$ *b)* $\lim_{x\to 0}\frac{x^2}{x^2+1} = 0,$ *c)* $\lim_{x\to 3}(x^2) = 9$

d) $\lim_{x\to -3}(x^2-9) = 0,$ *e)* $\lim_{x\to 0}(x^2) = 0$ *f)* $\lim_{x\to a}(-bx+c) - ab + c$

1.1.1 Basic limit theorems

Theorem 1.1

Operational Identities for Limits Suppose that $\lim_{x\to c} f(x) = L$ *and* $\lim_{x\to c} g(x) = M$ *and that* k *is constant. Then*

- $\lim_{x\to c} k \cdot f(x) = k \cdot \lim_{x\to c} f(x) = k \cdot L$
- $\lim_{x\to c}\Big[f(x) + g(x)\Big] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x) = L + M$
- $\lim_{x\to c}\Big[f(x) - g(x)\Big] = \lim_{x\to c} f(x) - \lim_{x\to c} g(x) = L - M$
- $\lim_{x\to c}\Big[f(x) \cdot g(x)\Big] = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x) = L \cdot M$
- $\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \frac{L}{M}$ *provided* $M \neq 0$

1.1.2 Direct Substitution Property

If $f(x)$ is a polynomial or a rational function and a is in the domain of $f(x)$, then

$$\lim_{x \to a} f(x) = f(a)$$

Example 1.3

1. *Evaluate the following limits and justify each step.*

   (a) $\lim_{x \to 5} 2x^2 - 3x + 4$

   (b) $\lim_{x \to 4} \sqrt[3]{\dfrac{x}{-7x+1}}$

Solution:

1. (a)

$$\begin{aligned}
\lim_{x \to 5}(2x^2 - 3x + 4) &= \lim_{x \to 5}(2x^2) - \lim_{x \to 5}(-3x) + 4 \quad \textit{(by rule 1 and 2)} \\
&= 2\lim_{x \to 5} x^2 - 3\lim_{x \to 5} x + 4 \quad \textit{(by rule 3 and 7)} \\
&= 2(5^2) - 3(5) + 4 \quad \textit{(by rule 9)} \\
&= 50 - 15 + 4 \\
&= 39
\end{aligned}$$

   (b)

$$\begin{aligned}
\lim_{x \to 4} \sqrt[3]{\frac{x}{-7x+1}} &= \sqrt[3]{\lim_{x \to 4}[\frac{x}{-7x+1}]} \quad \textit{(by low 11)} \\
&= \sqrt[3]{\frac{\lim_{x \to 4} x}{\lim_{x \to 4}(-7x+1)}} \quad \textit{(by low 5)} \\
&= \sqrt[3]{\frac{4}{-7(4)+1}} \quad \textit{(by low 3,9 and 1)} \\
&= \sqrt[3]{\frac{4}{-27}} \\
&= -\frac{\sqrt[3]{4}}{3}
\end{aligned}$$

Theorem 1.2

If $f(x) \leqslant g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a, then

$$\lim_{x \to a} f(x) \leqslant \lim_{x \to a} g(x)$$

Theorem 1.3: Squeeze Theorem

If $f(x) \leqslant g(x) \leqslant h(x)$ when x is near a (except possibly at a) and

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L \quad \text{then} \quad \lim_{x\to a} g(x) = L$$

which is sometimes called the Sandwich Theorem.

Example 1.4: S

ow that $\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$
Solution:*First note that we cannot use*

$$\lim_{x\to 0} x^2 \sin\frac{1}{x} = \lim_{x\to 0} x^2 \cdot \lim_{x\to 0} \sin\frac{1}{x}$$

because $\lim_{x\to 0} \sin\frac{1}{x}$ *does not exist. However, since*

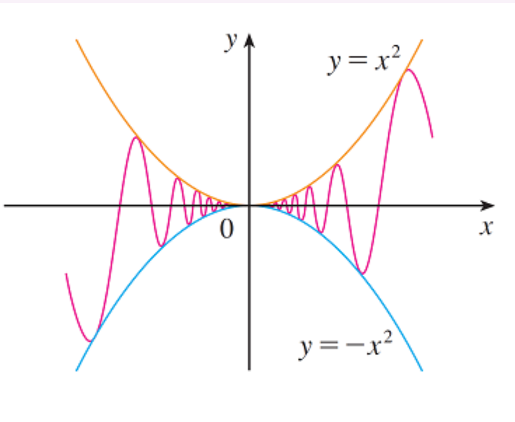
$$-1 \leq \sin\frac{1}{x} \leq 1$$

Multiply both side by x^2 we get

$$-x^2 \leq x^2 \sin\frac{1}{x} \leq x^2$$

We know that $\lim_{x\to 0} -x^2 = 0$ *and* $\lim_{x\to 0} x^2 = 0$ *taking* $f(x) = -x^2$, $h(x) = x^2$ *and* $g(x) = x^2 \sin\frac{1}{x}$ *by the Squeeze theorem, we obtain*

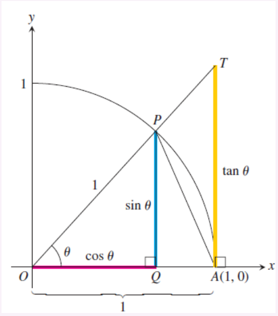
$$\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$$



Example 1.5

Show that
$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$$

Proof: *To show that the limit is 1, we begin with positive values of θ less than π. Notice that from the following figure*



We have Area $\triangle OAP \leqslant$ Area sector $OAP \leqslant$ Area $\triangle OAT$. We can express these areas in terms of θ as follows:

$$\begin{aligned}
\text{Area } \triangle OAP &= \frac{1}{2}\text{base}\times\text{hieght} = \frac{1}{2}(1)(\sin\theta)=\frac{\sin\theta}{2}\\
\text{Area sector } OAP &= \frac{1}{2}r^2\theta=\frac{1}{2}(1^2)(\theta)=\frac{\theta}{2}\\
\text{area } \triangle OAT &= \frac{1}{2}\text{base}\times\text{hieght} = \frac{1}{2}(1)(\tan\theta)=\frac{\tan\theta}{2}
\end{aligned}$$

Thus $\dfrac{\sin\theta}{2}\leqslant\dfrac{\theta}{2}\leqslant\dfrac{\tan\theta}{2}$.
This last inequality goes the same way if we divide all three terms by the number $\frac{2}{\sin\theta}$ *which is positive since* $0<\theta<\pi/2$

$$1\leqslant\frac{\theta}{\sin\theta}\leqslant\frac{1}{\cos\theta}$$

Taking reciprocals reverses the inequalities:

$$1\geqslant\frac{\sin\theta}{\theta}\geqslant\frac{\cos\theta}{1}$$

Since $\lim\limits_{\theta\to 0}\dfrac{\cos\theta}{1}=1$ *and* $\lim\limits_{\theta\to 0}1=1$ *then by the Sandwich theorem gives*

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$$

Recall that $\sin\theta$ and θ are both odd functions . Therefore, $f(\theta)=\dfrac{\sin\theta}{\theta}$ *is an even function, with a graph symmetric about the y-axis (see the above figure). This symmetry implies that the left-hand limit at 0 exists and has the same value as the right-hand limit:*

$$\lim_{\theta\to 0^+}\frac{\sin\theta}{\theta}=1=\lim_{\theta\to 0^-}\frac{\sin\theta}{\theta}$$

So $\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta}=1$

1.1.3 One sided limits

Definition 1.3

Let f be a function which is defined at every number in some open interval (a, c). Then the limit of $f(x)$, as x approaches from the right a, is L, written

$$\lim_{x \to a^+} f(x) = L$$

if for any $\epsilon > 0$, however small, there exists a $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < x - a < \delta$

Example 1.6

using the definition to prove that $\lim\limits_{x \to 0^+} \sqrt{x} = 0$

Solution:

1. *Guessing a value for δ. Let ϵ be a given positive number. Here $a = 0$ and $L = 0$ so we want to find a number δ such that $|\sqrt{x} - 0| < \epsilon$ whenever $0 < x - 0 < \delta$ that is $\sqrt{x} < \epsilon$ whenever $0 < x < \delta$ or, squaring both sides of the inequality $\sqrt{x} < \epsilon$, we get $x < \epsilon^2$ whenever $0 < x < \delta$. This suggests that we should choose $\delta = \epsilon$*

2. *Showing that this δ works. Given $\epsilon > 0$, let $\delta = \epsilon^2$. If $0 < x < \delta$, then $\sqrt{x} < \sqrt{\delta} = \sqrt{\epsilon^2} = \epsilon$. So $|\sqrt{x} - 0| < \epsilon$. According to the definition, this shows that* $\lim_{x \to 0^+} \sqrt{x} = 0$

Definition 1.4

Let f be a function which is defined at every number in some open interval (d, a). Then the limit of $f(x)$, as x approaches from the left a, is L, written

$$\lim_{x \to a^-} f(x) = L$$

if for any $\epsilon > 0$, however small, there exists a $\delta > 0$ such that
$|f(x) - L| < \epsilon$ whenever $-\delta < x - a < 0$

Example 1.7

Show that $\lim\limits_{x \to 4^-} \sqrt{4 - x} = 0$

Solution: *For every $\epsilon > 0$ we need to find a $\delta > 0$ such that if $-\delta < x - 4 < 0$ then $|\sqrt{4 - x} - 0| < \epsilon$. Since $|\sqrt{4 - x} - 0| = |\sqrt{4 - x}| < \epsilon \Rightarrow |4 - x| < \epsilon^2 \Rightarrow -\epsilon^2 < (x - 4) < \epsilon^2 \Rightarrow -\epsilon^2 < (x - 4) < 0$ then if $-\delta < (x - 4) < 0$ then $-\epsilon^2 < (x - 4) < 0$. This suggests that we should choose $\delta = \epsilon^2$. Therefore, we conclude that*

$$\lim_{x \to 4^-} \sqrt{4 - x} = 0$$

Theorem 1.4

$\lim\limits_{x \to c} f(x) = L$ *if and only if*

$$\lim_{x \to c^+} f(x) = L = \lim_{x \to c^-} f(x)$$

Exercise 1.2

Show that a) $\lim\limits_{x\to 0} |x| = 0$ *and* $\lim_{x\to 0} \dfrac{|x|}{x}$ *does not exist.*

1.1.4 Infinite limits, limit at infinity and asymptotes

Definition 1.5

Let $f(x)$ be a function defined on some open interval that contains the number a, except possibly at a itself. Then

i $\lim\limits_{x\to a^+} f(x) = \infty$ *means that for every positive number M there is a positive number δ such that $f(x) > M$ whenever $0 < x - a < \delta$*

ii $\lim\limits_{x\to a^-} f(x) = \infty$ *means that for every positive number M there is a positive number δ such that $f(x) > M$ whenever $-\delta < x - a < 0$*

iii $\lim\limits_{x\to a} f(x) = \infty$ *means that for every positive number M there is a positive number δ such that $f(x) > M$ whenever $|x - a| < \delta$*

Example 1.8

By using the definition prove that $\lim_{x\to 0} \dfrac{1}{x^2} = \infty$

Solution: *Let M be given large number, we want to find a $\delta > 0$ such that $f(x) > M$ when ever $0 < |x - 0| < \delta$ or $\dfrac{1}{x^2} > M$ when ever $0 < |x| < \delta$ that is, $x^2 < \dfrac{1}{M}$ when ever $0 < |x| < \delta$ or $|x| < \dfrac{1}{\sqrt{M}}$ when ever $0 < |x| < \delta$. This suggests that we should take $\delta = \dfrac{1}{\sqrt{M}}$. Therefore* $\lim\limits_{x\to 0} \dfrac{1}{x^2} = \infty$.

Definition 1.6

Let $f(x)$ be a function defined on some open interval that contains the number a, except possibly at a itself. Then

i $\lim\limits_{x\to a^+} f(x) = -\infty$ *means that for every negative number N there is a positive number δ such that $f(x) < N$ whenever $0 < x - a < \delta$*

ii $\lim\limits_{x\to a^-} f(x) = -\infty$ *means that for every negative number N there is a positive number δ such that $f(x) < N$ whenever $-\delta < x - a < 0$*

iii $\lim\limits_{x\to a} f(x) = -\infty$ *means that for every negative number N there is a positive number δ such that $f(x) < N$ whenever $0 < |x - a| < \delta$*

Example 1.9

Show that $\lim_{x\to 0^-} \frac{1}{x} = -\infty$

Solution: *Let* N *be given negative small number, we want to find a* $\delta > 0$ *such that* $f(x) < N$ *when ever* $-\delta < x - 0 < 0$ *or* $\frac{1}{x} < N$ *when ever* $-\delta < x - 0 < 0$ *that is,* $x > \frac{1}{N}$ *when ever* $-\delta < x < 0$. *This suggests that we should take* $\delta = -\frac{1}{N}$. *Therefore* $\lim_{x\to 0^-} \frac{1}{x} = -\infty$.

Definition 1.7

We say that $f(x)$ *has the limit* L *as x approaches infinity and write*

$$\lim_{x\to\infty} f(x) = L$$

if, for every number $\epsilon > 0$ *there exists a corresponding positive number* M *such that for all* x

$$x > M \Rightarrow |f(x) - L| < \epsilon$$

Example 1.10

Show that

$$\lim_{x\to\infty} \frac{1}{x} = 0$$

Solution: *Let* $\epsilon > 0$ *be given. We must find a positive number* M *such that for all* x

$$x > M \Rightarrow |\frac{1}{x} - 0| = |\frac{1}{x}| < \epsilon$$

$x > M \Rightarrow x > \frac{1}{\epsilon}$ *because* x *is positive . The implication will hold if* $M = \frac{1}{\epsilon}$ *or any larger positive number. This proves that*

$$\lim_{x\to\infty} \frac{1}{x} = 0$$

Definition 1.8

We say that $f(x)$ *has the limit* L *as x approaches negative infinity and write*
if, for every number $\epsilon > 0$ *there exists a corresponding negative number* N *such that for all x*

$$x < N \quad \Rightarrow \quad |f(x) - L| < \epsilon$$

Example 1.11

Show that

$$\lim_{x\to-\infty} \frac{1}{x} = 0$$

Solution: *Let $\epsilon > 0$ be given. We must find a negative number N such that for all x*

$$x < N \Rightarrow |\frac{1}{x} - 0| = |\frac{1}{x}| < \epsilon$$

$x < N \Rightarrow x < -\frac{1}{\epsilon}$ because x is negative. The implication will hold if $N = -\frac{1}{\epsilon}$ or any small number. This proves that

$$\lim_{x\to-\infty} \frac{1}{x} = 0$$

Theorem 1.5

Suppose that c is a constant and the limits $\lim\limits_{x\to\pm\infty} f(x) = L$ and $\lim\limits_{x\to\pm\infty} g(x) = M$ exist. Then

1. *Sum rule* $\lim\limits_{x\to\pm\infty}[f(x) + g(x)] = \lim\limits_{x\to\pm\infty} f(x) + \lim\limits_{x\to\pm\infty} g(x) = L + M.$
2. *Difference rule* $\lim\limits_{x\to\pm\infty}[f(x) - g(x)] = \lim\limits_{x\to\pm\infty} f(x) - \lim\limits_{x\to\pm\infty} g(x) = L - M.$
3. *Constant Rule* $\lim\limits_{x\to\pm\infty}[cf(x)] = c\lim\limits_{x\to\pm\infty} f(x) = cL.$
4. *Product Rule* $\lim\limits_{x\to\pm\infty}[f(x)g(x)] = \lim\limits_{x\to\pm\infty} f(x)\lim\limits_{x\to\pm\infty} g(x) = LM.$
5. *Quotient Rule* $\lim\limits_{x\to\pm\infty}[\frac{f(x)}{g(x)}] = \frac{\lim_{x\to\pm\infty} f(x)}{\lim\limits_{x\to\pm\infty} g(x)} = \frac{L}{M}$ *if* $\lim\limits_{x\to\pm\infty} g(x) \neq 0.$

Example 1.12

Using the above theorem evaluate the following limits at infinity

a). $\lim\limits_{x\to\infty}\dfrac{\sqrt{4x^2+2}}{3x+1}$ b). $\lim\limits_{x\to\infty}\dfrac{x^2+1}{2x-3}$

Solution: a).

$$\begin{aligned}\lim_{x\to\infty}\frac{\sqrt{4x^2+2}}{3x+1} &= \lim_{x\to\infty}\frac{\sqrt{x^2(4+\frac{1}{x^2})}}{x(3+\frac{1}{x})}\\ &= \lim_{x\to\infty}\frac{\sqrt{(4+\frac{1}{x^2})}}{(3+\frac{1}{x})}\\ &= \frac{\sqrt{\lim\limits_{x\to\infty}(4+\frac{1}{x^2})}}{\lim\limits_{x\to\infty}(3+\frac{1}{x})}\\ &= \frac{\sqrt{(\lim\limits_{x\to\infty}4+\lim\limits_{x\to\infty}\frac{1}{x^2})}}{(\lim\limits_{x\to\infty}3+\lim\limits_{x\to\infty}\frac{1}{x})}\\ &= \frac{\sqrt{(4+0)}}{(3+0)}\\ &= \frac{2}{3}\end{aligned}$$

b).

$$\begin{aligned}\lim_{x\to\infty}\frac{x^2+1}{2x-3} &= \lim_{x\to\infty}\frac{x+\frac{1}{x}}{2-\frac{3}{x}}\\ &= \frac{\lim\limits_{x\to\infty}x+\lim\limits_{x\to\infty}\frac{1}{x}}{\lim\limits_{x\to\infty}2-\lim\limits_{x\to\infty}\frac{3}{x}}\\ &= \frac{\infty+0}{2-0}\\ &= \frac{\infty}{2}\\ &= \infty\end{aligned}$$

Definition 1.9

The line $y=b$ is called a horizontal asymptote of the curve $y=f(x)$ if either $\lim\limits_{x\to\infty}f(x)=b$ *or/and* $\lim\limits_{x\to-\infty}f(x)=b$

Example 1.13

Find $\lim\limits_{x\to\infty}\dfrac{1}{x}$ *and* $\lim\limits_{x\to-\infty}\dfrac{1}{x}$.

Solution: *Observe that when is large, is small. For instance, In fact, by taking large enough, we can make as close to* 0 *as we please. Therefore, according to Definition , we have Similar reasoning shows that when x is large negative, $1/x$.*

Definition 1.10

A line $x=a$ is a vertical asymptote of the graph of a function $y=f(x)$ if either $\lim\limits_{x\to a^+}f(x)=\pm\infty$ *or/and* $\lim\limits_{x\to a^-}f(x)=\pm\infty$

Example 1.14

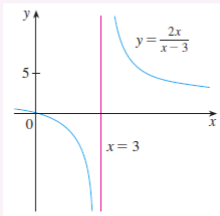
Find $\lim_{x\to 3^-}(\frac{2x}{x-3})$ *and* $\lim_{x\to 3^+}(\frac{2x}{x-3})$.
Solution: *If x is close to 3 but larger than 3, then the denominator $x-3$ is a small positive number and $2x$ is close to 6. So the quotient $2x/x-3$ is a large positive number.*

$$\lim_{x\to 3^+}(\frac{2x}{x-3}) = \infty$$

Likewise If x is close to 3 but smaller than 3, then the denominator $x-3$ is a small negative number and $2x$ is close to 6. So the quotient $2x/x-3$ is a large negative number.

$$\lim_{x\to 3^-}(\frac{2x}{x-3}) = -\infty$$

The graph of the curve $y = \frac{2x}{x-3}$ is given in the following figure. The line is a vertical asymptote.



Example 1.15

Find the vertical and horizontal asymptotes for the graph of

a). $f(x) = \frac{\sqrt{x^2+2}}{x-1}$ *b).* $f(x) = 2x + 1 - \sqrt{4x^2+5}$ *c).* $f(x) = \sqrt[3]{\frac{x^2+3}{27x^2-1}}$

Solution: *a) We are interested in the behavior as $x \to \pm\infty$ and as $x \to 1$ where the denominator is zero.*
and then

$$\lim_{x\to 1^-}\frac{\sqrt{x^2+2}}{x-1} = -\infty, \lim_{x\to 1^+}\frac{\sqrt{x^2+2}}{x-1} = \infty, \lim_{x\to\infty}\frac{\sqrt{x^2+2}}{x-1} = 1$$

and

$$\lim_{x\to -\infty}\frac{\sqrt{x^2+2}}{x-1} = -1$$

$\therefore$ *$x = 1$ is the vertical asymptote of $f(x) = \frac{\sqrt{x^2+2}}{x-1}$ and also $y = 1$ and $y = -1$ are horizontal asymptote of $f(x) = \frac{\sqrt{x^2+2}}{x-1}$.*

Special limit

- $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x}=0$
- $\lim\limits_{x\to+\infty}\left(1+\dfrac{k}{x}\right)^{mx}=e^{mk}$
- $\lim\limits_{x\to+\infty}\left(1+\dfrac{1}{x}\right)^{x}=e$
- $\lim\limits_{x\to+\infty}\left(1-\dfrac{1}{x}\right)^{x}=\dfrac{1}{e}$
- $\lim\limits_{x\to+\infty}\left(\dfrac{x}{x+k}\right)^{x}=\dfrac{1}{e^k}$
- $\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}}=e$
- $\lim\limits_{n\to\infty}\dfrac{n}{\sqrt[n]{n!}}=e$
- $\lim\limits_{n\to\infty}2^n\underbrace{\sqrt{2-\sqrt{2+\sqrt{2+...+\sqrt{2}}}}}_{n}=\pi$
- $\lim\limits_{x\to 0}\left(\dfrac{a^x-1}{x}\right)=\ln a, \qquad \forall\ a>0$
- $\lim\limits_{x\to 0}\left(1+a\left(e^{-x}-1\right)\right)^{-\frac{1}{x}}=e^a$

Exercise 1.3

Find the limit

1. $\lim\limits_{x\to 0}\frac{4x}{\sin 3x}$
2. $\lim\limits_{x\to 0}\frac{\cos 3x-\cos x}{x^2}$
3. $\lim\limits_{x\to 0}\frac{\sin 5x-\sin 3x}{\sin x}$
4. $\lim\limits_{x\to 0}\frac{\sin ax}{\sin bx}$
5. $\lim\limits_{x\to 0}\frac{\tan x-\sin x}{x^3}$
6. $\lim\limits_{x\to 0+0}\frac{\sqrt{1-\cos x}}{x}$
7. $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+5}$
8. $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^{3x}$
9. $\lim\limits_{x\to\infty}\left(1+\frac{6}{x}\right)^{x}$
10. $\lim\limits_{x\to 0}\sqrt[x]{1+3x}$
11. $\lim\limits_{x\to\infty}\left(\frac{x+a}{x-a}\right)^{x}$
12. $\lim\limits_{x\to\infty}\left(\frac{x}{x+1}\right)^{x}$.
13. $\lim\limits_{x\to\infty}\left(\frac{x+3}{x-2}\right)^{x-1}$
14. $\lim\limits_{x\to a}\frac{\ln x-\ln a}{x-a}$,
15. $\lim\limits_{x\to 0}(1+\sin x)^{\frac{1}{x}}$

Solution:

1.

$$\begin{aligned}L&=\lim_{x\to 0}\frac{4x}{\sin 3x}\\&=\lim_{x\to 0}\frac{3\cdot 4x}{3\sin 3x}\\&=\frac{4}{3}\lim_{x\to 0}\frac{3x}{\sin 3x}\\&=\frac{4}{3}\lim_{x\to 0}\frac{1}{\frac{\sin 3x}{3x}}\\&=\frac{4}{3}\frac{\lim\limits_{x\to 0}1}{\lim\limits_{x\to 0}\frac{\sin 3x}{3x}}.\end{aligned}$$

Since $3x\to 0$ as $x\to 0$, we can write:

$$\begin{aligned}L&=\frac{4}{3}\frac{\lim\limits_{x\to 0}1}{\lim\limits_{x\to 0}\frac{\sin 3x}{3x}}\\&=\frac{4}{3\lim\limits_{3x\to 0}\frac{\sin 3x}{3x}}\\&=\frac{4}{3\cdot 1}=\frac{4}{3}.\end{aligned}$$

2. We factor the numerator:

$$\begin{aligned}\cos 3x-\cos x&=-2\sin\frac{3x-x}{2}\sin\frac{3x+x}{2}\\&=-2\sin x\sin 2x.\end{aligned}$$

This yields

$$\begin{aligned}\lim_{x\to 0}\frac{\cos 3x-\cos x}{x^2}&=\lim_{x\to 0}\frac{(-2\sin x\sin 2x)}{x^2}\\&=-2\lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\frac{\sin 2x}{x}\\&=-2\cdot 1\cdot\lim_{2x\to 0}\frac{2\sin 2x}{2x}\\&=-2\cdot 2\lim_{2x\to 0}\frac{\sin 2x}{2x}=-4.\end{aligned}$$

3. We use the following trigonometric identity:

$$\sin x-\sin y=2\sin\frac{x-y}{2}\cos\frac{x+y}{2}.$$

Then we obtain

$$\begin{aligned}\lim_{x\to 0}\frac{\sin 5x-\sin 3x}{\sin x} &= \lim_{x\to 0}\frac{2\sin\frac{5x-3x}{2}\cos\frac{5x+3x}{2}}{\sin x}\\ &= \lim_{x\to 0}\frac{2\sin x\cos 4x}{\sin x}\\ &= \lim_{x\to 0}\left(2\cos 4x\right).\end{aligned}$$

As $\cos 4x$ is a continuous function at $x=0$, then

$$\begin{aligned}\lim_{x\to 0}\left(2\cos 4x\right) &= 2\lim_{x\to 0}\cos 4x\\ &= 2\cdot\cos\left(4\cdot 0\right)=2\cdot 1=2.\end{aligned}$$

4.

$$\begin{aligned}L &= \lim_{x\to 0}\frac{\sin ax}{\sin bx}\\ &= \lim_{x\to 0}\left(\frac{\sin ax}{\sin bx}\cdot\frac{a}{b}\cdot\frac{bx}{ax}\right)\\ &= \lim_{x\to 0}\left(\frac{\sin ax}{ax}\cdot\frac{bx}{\sin bx}\cdot\frac{a}{b}\right)\\ &= \frac{a}{b}\frac{\lim_{x\to 0}\frac{\sin ax}{ax}}{\lim_{x\to 0}\frac{\sin bx}{bx}}.\end{aligned}$$

Obviously, $ax\to 0$ and $bx\to 0$ as $x\to 0$. Then

$$L=\frac{a}{b}\frac{\lim_{x\to 0}\frac{\sin ax}{ax}}{\lim_{x\to 0}\frac{\sin bx}{bx}}=\frac{a}{b}\cdot\frac{1}{1}=\frac{a}{b}.$$

5. We apply the following transformations:

$$\begin{aligned}L &= \lim_{x\to 0}\frac{\tan x-\sin x}{x^3}\\ &= \lim_{x\to 0}\frac{\frac{\sin x}{\cos x}-\sin x}{x^3}\\ &= \lim_{x\to 0}\frac{\sin x\left(\frac{1}{\cos x}-1\right)}{x^3}\\ &= \lim_{x\to 0}\frac{\sin x\left(1-\cos x\right)}{x^3\cos x}\\ &= \lim_{x\to 0}\left[\frac{\sin x}{x}\cdot\frac{1-\cos x}{x^2\cos x}\right].\end{aligned}$$

As $1-\cos x=2\sin^2\frac{x}{2}$, we have

$$\begin{aligned}L &= \lim_{x\to 0}\left[\frac{\sin x}{x}\cdot\frac{1-\cos x}{x^2\cos x}\right]\\ &= \lim_{x\to 0}\left[\frac{\sin x}{x}\cdot\frac{2\sin^2\frac{x}{2}}{x^2\cos x}\right]\\ &= \frac{\lim_{x\to 0}\frac{\sin x}{x}\cdot 2\lim_{x\to 0}\frac{\sin^2\frac{x}{2}}{x^2}}{\lim_{x\to 0}\cos x}.\end{aligned}$$

Here

$$\lim_{x\to 0}\frac{\sin x}{x}=1 \text{ and } \lim_{x\to 0}\cos x=1.$$

Hence,

$$\begin{aligned}L &= 2\lim_{x\to 0}\frac{\sin^2\frac{x}{2}}{x^2}\\ &= 2\lim_{x\to 0}\left(\frac{\sin^2\frac{x}{2}}{x^2}\cdot\frac{4}{4}\right)\\ &= 2\lim_{x\to 0}\left(\frac{\sin^2\frac{x}{2}}{\frac{x^2}{4}}\cdot\frac{1}{4}\right)\\ &= \frac{1}{2}\lim_{x\to 0}\frac{\sin^2\frac{x}{2}}{\left(\frac{x}{2}\right)^2}.\end{aligned}$$

Here $\frac{x}{2}\to 0$ when $x\to 0$, therefore,

$$L=\frac{1}{2}\lim_{\frac{x}{2}\to 0}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2=\frac{1}{2}\cdot 1^2=\frac{1}{2}.$$

6. Use the trigonometric formula:

$$1-\cos x=2\sin^2\frac{x}{2}.$$

Then the limit can be written in the form

$$\begin{aligned}L &= \lim_{x\to 0+0}\frac{\sqrt{1-\cos x}}{x}\\ &= \lim_{x\to 0+0}\frac{\sqrt{2\sin^2\frac{x}{2}}}{x}\\ &= \sqrt{2}\lim_{x\to 0+0}\frac{\sqrt{\sin^2\frac{x}{2}}}{x}\\ &= \sqrt{2}\lim_{x\to 0+0}\sqrt{\frac{\sin^2\frac{x}{2}}{x^2}}\\ &= \sqrt{2}\lim_{x\to 0+0}\sqrt{\frac{\sin^2\frac{x}{2}}{x^2}\cdot\frac{4}{4}}\\ &= \sqrt{2}\lim_{x\to 0+0}\left(\sqrt{\frac{\sin^2\frac{x}{2}}{\frac{x^2}{4}}}\cdot\frac{1}{\sqrt{4}}\right)\\ &= \frac{\sqrt{2}}{2}\lim_{x\to 0+0}\sqrt{\frac{\sin^2\frac{x}{2}}{\left(\frac{x}{2}\right)^2}}\\ &= \frac{1}{\sqrt{2}}\sqrt{\lim_{x\to 0+0}\frac{\sin^2\frac{x}{2}}{\left(\frac{x}{2}\right)^2}}\\ &= \frac{1}{\sqrt{2}}\sqrt{\left[\lim_{\frac{x}{2}\to 0+0}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)\right]^2}\\ &= \frac{1}{\sqrt{2}}\cdot\sqrt{1^2}=\frac{1}{\sqrt{2}}.\end{aligned}$$

We used here the fact that the limit remains the same when replacing $x \to 0$ to $\frac{x}{2} \to 0$

7.

$$\begin{aligned}
&\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+5}\\
&=\lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^{n}\left(1+\frac{1}{n}\right)^{5}\right]\\
&=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}\cdot\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{5}\\
&=e\cdot 1=e.
\end{aligned}$$

8. By the product rule for limits, we obtain

$$\begin{aligned}
&\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{3x}\\
&=\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}\cdot\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}\cdot\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}\\
&=e\cdot e\cdot e=e^3.
\end{aligned}$$

9. Substituting $\frac{6}{x}=\frac{1}{y}$, so that $x=6y$ and $y\to\infty$ as $x\to\infty$, we obtain

$$\begin{aligned}
&\lim_{x\to\infty}\left(1+\frac{6}{x}\right)^{x}\\
&=\lim_{y\to\infty}\left(1+\frac{1}{y}\right)^{6y}\\
&=\lim_{y\to\infty}\left[\left(1+\frac{1}{y}\right)^{y}\right]^{6}\\
&=\left[\lim_{y\to\infty}\left(1+\frac{1}{y}\right)^{y}\right]^{6}=e^6.
\end{aligned}$$

10.

$$\begin{aligned}
&\lim_{x\to0}\sqrt[x]{1+3x}=\lim_{x\to0}(1+3x)^{\frac{1}{x}}\\
&=\lim_{3x\to0}(1+3x)^{\frac{1}{3x}\cdot3}=\lim_{3x\to0}\left[(1+3x)^{\frac{1}{3x}}\right]^{3}\\
&=\left[\lim_{3x\to0}(1+3x)^{\frac{1}{3x}}\right]^{3}=e^3.
\end{aligned}$$

11. We first transform the base of the function:

$$\begin{aligned}
L&=\lim_{x\to\infty}\left(\frac{x+a}{x-a}\right)^{x}\\
&=\lim_{x\to\infty}\left(\frac{x-a+2a}{x-a}\right)^{x}\\
&=\lim_{x\to\infty}\left(1+\frac{2a}{x-a}\right)^{x}.
\end{aligned}$$

Introduce the new variable: $y=\frac{2a}{x-a}$. As $x\to\infty, y\to0$ and, hence,

$$x-a=\frac{2a}{y},\quad x=a+\frac{2a}{y}.$$

Substituting this into the function gives

$$\begin{aligned}
L&=\lim_{x\to\infty}\left(1+\frac{2a}{x-a}\right)^{x}\\
&=\lim_{y\to0}(1+y)^{a+\frac{2a}{y}}\\
&=\lim_{y\to0}(1+y)^{a}\cdot\lim_{y\to0}(1+y)^{\frac{2a}{y}}\\
&=1\cdot e^{2a}=e^{2a}.
\end{aligned}$$

12. First we transform the base:

$$\begin{aligned}
L&=\lim_{x\to\infty}\left(\frac{x}{x+1}\right)^{x}\\
&=\lim_{x\to\infty}\left(\frac{x+1-1}{x+1}\right)^{x}\\
&=\lim_{x\to\infty}\left(1-\frac{1}{x+1}\right)^{x}.
\end{aligned}$$

Let $-\frac{1}{x+1}=y$. Then $x+1=-\frac{1}{y}$, $\Rightarrow$ $x=-\frac{1}{y}-1$ and $y\to0$, if $x\to\infty$. Now we can find the limit:

$$\begin{aligned}
L&=\lim_{x\to\infty}\left(1-\frac{1}{x+1}\right)^{x}\\
&=\lim_{y\to0}(1+y)^{-\frac{1}{y}-1}\\
&=\frac{\lim_{y\to0}(1+y)^{-\frac{1}{y}}}{\lim_{y\to0}(1+y)^{+1}}\\
&=\frac{\lim_{y\to0}\left[(1+y)^{\frac{1}{y}}\right]^{-1}}{1}\\
&=\left[\lim_{y\to0}(1+y)^{\frac{1}{y}}\right]^{-1}=\frac{1}{e}.
\end{aligned}$$

13. We can transform this limit as follows:

$$\begin{aligned}
L&=\lim_{x\to\infty}\left(\frac{x+3}{x-2}\right)^{x-1}\\
&=\lim_{x\to\infty}\left(\frac{x-2+5}{x-2}\right)^{x-1}\\
&=\lim_{x\to\infty}\left(1+\frac{5}{x-2}\right)^{x-1}\\
&=\lim_{x\to\infty}\left[\left(1+\frac{5}{x-2}\right)^{\frac{x-2}{5}}\right]^{\frac{5(x-1)}{x-2}}.
\end{aligned}$$

Replace the variable:
$$\frac{5}{x-2} = y, \quad \Rightarrow x-2 = \frac{5}{y}, \quad \Rightarrow x = \frac{5}{y} + 2.$$
Here $y \to 0$ as $x \to 0$. Then the limit is

$$\begin{aligned} L &= \lim_{x\to\infty} \left[\left(1 + \frac{5}{x-2}\right)^{\frac{x-2}{5}} \right]^{\frac{5(x-1)}{x-2}} \\ &= \lim_{y\to 0} \left[(1+y)^{\frac{1}{y}} \right]^{y\left(\frac{5}{y}+2-1\right)} \\ &= \lim_{y\to 0} \left[(1+y)^{\frac{1}{y}} \right]^{5+y} \\ &= \lim_{y\to 0} \left[(1+y)^{\frac{1}{y}} \right]^{5} \cdot \lim_{y\to 0} \left[(1+y)^{\frac{1}{y}} \right]^{y} \\ &= \lim_{y\to 0} \left[(1+y)^{\frac{1}{y}} \right]^{5} \cdot \lim_{y\to 0} (1+y) \\ &= e^5 \cdot 1 = e^5. \end{aligned}$$

14. Let $x-a = t$. It is easy to see that $t \to 0$ as $x \to a$. Then
$$\begin{aligned} L &= \lim_{x\to a} \frac{\ln x - \ln a}{x-a} \\ &= \lim_{t\to 0} \frac{\ln(t+a) - \ln a}{t} \\ &= \lim_{t\to 0} \frac{\ln \frac{t+a}{a}}{t} \\ &= \lim_{t\to 0} \frac{1}{t} \ln\left(1 + \frac{t}{a}\right). \end{aligned}$$
Make one more change of variable:
$$\frac{t}{a} = z, \quad z \to 0 \text{ as } t \to 0.$$
Hence, the limit becomes
$$\begin{aligned} L &= \lim_{t\to 0} \frac{1}{t} \ln\left(1 + \frac{t}{a}\right) \\ &= \lim_{z\to 0} \frac{1}{az} \ln(1+z) \\ &= \frac{1}{a} \lim_{z\to 0} \ln(1+z)^{\frac{1}{z}} \\ &= \frac{1}{a} \ln\left[\lim_{z\to 0} (1+z)^{\frac{1}{z}}\right] \\ &= \frac{1}{a} \ln e = \frac{1}{a}. \end{aligned}$$

15. The limit can be represented in the following form:
$$\begin{aligned} L &= \lim_{x\to 0} (1+\sin x)^{\frac{1}{x}} \\ &= \lim_{x\to 0} (1+\sin x)^{\frac{1}{\sin x}\cdot\frac{\sin x}{x}} \\ &= \lim_{x\to 0} \left[(1+\sin x)^{\frac{1}{\sin x}}\right]^{\frac{\sin x}{x}}. \end{aligned}$$
After taking logarithm, we have
$$\begin{aligned} \ln L &= \ln\left(\lim_{x\to 0} \left[(1+\sin x)^{\frac{1}{\sin x}}\right]^{\frac{\sin x}{x}}\right) \\ &= \lim_{x\to 0} \left(\frac{\sin x}{x} \ln\left[(1+\sin x)^{\frac{1}{\sin x}}\right]\right) \\ &= \lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0} \left(\ln\left[(1+\sin x)^{\frac{1}{\sin x}}\right]\right). \end{aligned}$$
We notice that $\lim_{x\to 0} \frac{\sin x}{x} = 1$. Besides that, $\sin x \to 0$ as $x \to 0$, therefore, we can replace the transition $x \to 0$ in the second limit with the equivalent limit $\sin x \to 0$. This yields
$$\begin{aligned} \ln L &= 1 \cdot \lim_{\sin x\to 0} \left(\ln\left[(1+\sin x)^{\frac{1}{\sin x}}\right]\right) \\ &= \ln \lim_{\sin x\to 0} \left[(1+\sin x)^{\frac{1}{\sin x}}\right]. \end{aligned}$$
As $\lim_{\sin x\to 0} (1+\sin x)^{\frac{1}{\sin x}} = e$
$$\ln L = \ln e = 1.$$
Thus, $L = e$

1.2 Continuity and One Sided Continuity

We are now ready to define the concept of a function being continuous. The idea is that we want to say that a function is continuous if you can draw its graph without taking your pencil off the page. But sometimes this will be true for some parts of a graph but not for others. Therefore, we want to start by defining what it means for a function to be continuous at one point. The definition is simple, now that we have the concept of limits:

Definition 1.11: (continuity at a point)

If $f(x)$ is defined on an open interval containing c, then $f(x)$ is said to be continuous at c if and only if

$$\lim_{x \to c} f(x) = f(c).$$

Note that for f to be continuous at c, the definition in effect requires three conditions:

1. that f is defined at c , so $f(c)$ exists,
2. the limit as x approaches c exists, and
3. the limit and $f(c)$ are equal.

If any of these do not hold then f is not continuous at c.

The idea of the definition is that the point of the graph corresponding to c will be close to the points of the graph corresponding to nearby x-values. Now we can define what it means for a function to be continuous in general, not just at one point.

Definition 1.12

A function is said to be continuous on (a, b) if it is continuous at every point of the interval (a, b) .

We often use the phrase "the function is continuous" to mean that the function is continuous at every real number. This would be the same as saying the function was continuous on $(-\infty, \infty)$, but it is a bit more convenient to simply say "continuous".

Note that, by what we already know, the limit of a rational, exponential, trigonometric or logarithmic function at a point is just its value at that point, so long as it's defined there. So, all such functions are continuous wherever they're defined. (Of course, they can't be continuous where they're not defined!)

1.2.1 Discontinuities

A discontinuity is a point where a function is not continuous. There are lots of possible ways this could happen, of course. Here we'll just discuss two simple ways.

1.2.2 Removable discontinuities

The function $f(x) = \dfrac{x^2 - 9}{x - 3}$ is not continuous at $x = 3$. It is discontinuous at that point because the fraction then becomes $\dfrac{0}{0}$, which is undefined. Therefore the function fails the first of our three conditions for continuity at the point $3, 3$ is just not in its domain.

However, we say that this discontinuity is removable. This is because, if we modify the function at that point, we can eliminate the discontinuity and make the function continuous. To see how to make the function $f(x)$ continuous, we have to simplify $f(x)$, getting $f(x) = \dfrac{x^2 - 9}{x - 3} = \dfrac{(x + 3)(x - 3)}{(x - 3)} = \dfrac{x + 3}{1} \cdot \dfrac{x - 3}{x - 3}$. We can define a new function $g(x))$ where

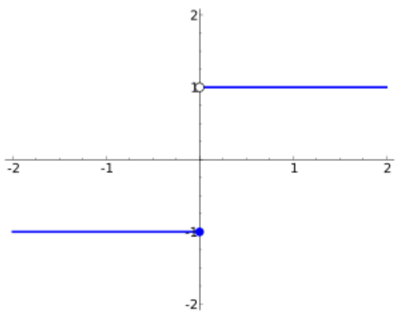
$g(x) = x + 3$. Note that the function $g(x))$ is not the same as the original function $f(x)$, because $g(x)$ is defined at $x = 3$, while $f(x)$ is not. Thus, $g(x)$ is continuous at $x = 3$, since $\lim_{x \to 3}(x + 3) = 6 = g(3)$. However, whenever $x \neq 3$, $f(x) = g(x)$; all we did to f to get g was to make it defined at $x = 3$.

In fact, this kind of simplification is often possible with a discontinuity in a rational function. We can divide the numerator and the denominator by a common factor (in our example $x - 3$ to get a function which is the same except where that common factor was 0 (in our example at $x = 3$. This new function will be identical to the old except for being defined at new points where previously we had division by 0.

However, this is not possible in every case. For example, the function $f(x) = \dfrac{x-3}{x^2 - 6x + 9}$ has a common factor of $x - 3$ in both the numerator and denominator, but when you simplify you are left with $g(x) = \dfrac{1}{x-3}$, which is still not defined at $x = 3$. In this case the domain of $f(x)$ and $g(x)$ are the same, and they are equal everywhere they are defined, so they are in fact the same function. The reason that $g(x)$ differed from $f(x)$ in the first example was because we could take it to have a larger domain and not simply that the formulas defining $f(x)$ and $g(x)$ were different.

1.2.3 Jump discontinuities

Illustration of a jump discontinuity



Not all discontinuities can be removed from a function. Consider this function: ot all discontinuities can be removed from a function. Consider this function: $k(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x \leq 0 \end{cases}$ Since $\lim_{x \to 0} k(x)$ does not exist, there is no way to redefine k at one point so that it will be continuous at 0. These sorts of discontinuities are called nonremovable discontinuities.

Note, however, that both one-sided limits exist; $\lim_{x \to 0^-} k(x) = -1$ and $\lim_{x \to 0^+} k(x) = 1$. The problem is that they are not equal, so the graph "jumps" from one side of 0 to the other. In such a case, we say the function has a jump discontinuity. (Note that a jump discontinuity is a kind of nonremovable discontinuity.)

One-Sided Continuity

Just as a function can have a one-sided limit, a function can be continuous from a particular side. For a function to be continuous at a point from a given side, we need the following three conditions:

1. the function is defined at the point,

2. the function has a limit from that side at that point and
3. the one-sided limit equals the value of the function at the point.

A function $f(x)$ is

- *Left-continuous* at $x = c$ if $\lim_{x \to c^-} f(x) = f(c)$
- *Right-continuous* at $x = c$ if $\lim_{x \to c^+} f(x) = f(c)$.

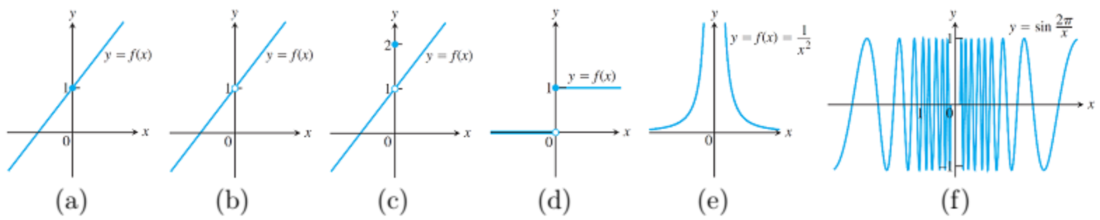
A function will be continuous at a point if and only if it is continuous from both sides at that point. Now we can define what it means for a function to be continuous on a closed interval.

Definition 1.13

(continuity on a closed interval) A function is said to be continuous on $[a, b]$ if and only if

1. *it is continuous on (a, b),*
2. *it is continuous from the right at a and*
3. *it is continuous from the left at b.*

Notice that, if a function is continuous, then it is continuous on every closed interval contained in its domain.



REMARK: The discontinuities in parts (b) and (c) are called removable discontinuities because we could remove them by redefining f at just the single number 0. The discontinuity in part (d) is called jump discontinuity because the function ”jumps” from one value to another. The discontinuities in parts (e) and (f) are called infinite or essential discontinuities.

Exercise 1.4

If the function $f(x) = \begin{cases} \cos(2\pi x - a), & x \leq -1 \\ x^3 + 1, & x \geq -1 \end{cases}$ *is continuous, what is the value of a?*

Solution:

We calculate the left-hand and right-hand limits at $x = -1$.

$$\lim_{x \to -1-0} f(x) = \lim_{x \to -1-0} \cos(2\pi x - a)$$
$$= \cos(-2\pi - a) = \cos a,$$

The function will be continuous at $x = -1$, if

$$\lim_{x \to -1-0} f(x) = \lim_{x \to -1+0} f(x) \quad \text{or} \quad \cos a = 0.$$

Hence,

$$a = \frac{\pi}{2} + \pi n, \quad n \in \mathbb{Z}.$$

Exercise 1.5

Let $f(x) = \begin{cases} x^2+2, & x \le 0 \\ ax+b, & 0 \le x \le 1 \\ 3+2x-x^2, & x \ge 1 \end{cases}$ *Determine* a *and* b *so that the function* $f(x)$ *is continuous everywhere.*

Solution: *Solution. The left-side limit at* $x = 0$

$$\lim_{x \to 0-0} f(x) = \lim_{x \to 0-0} \left(x^2+2\right) = 2.$$

is Then the value of $ax+b$ *at* $x = 0$ *must be equal to* 2.

$$ax+b = 2, \quad \Rightarrow a \cdot 0 + b = 2, \quad \Rightarrow b = 2.$$

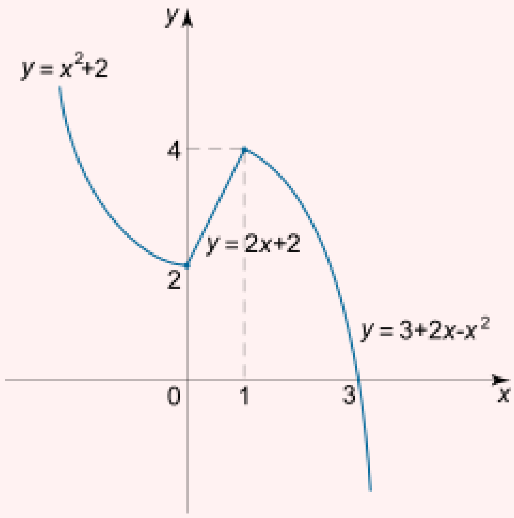
Similarly, the right-side limit at $x = 1$ *is*

$$\lim_{x \to 1+0} f(x)$$
$$= \lim_{x \to 1+0} \left(3+2x-x^2\right) \qquad = 3+2-1 = 4.$$

As seen, the value of $ax+2$ *at* $x = 1$ *must be equal to* 4.

$$ax+2 = 4, \quad \Rightarrow a \cdot 1 + 2 = 4, \quad \Rightarrow a = 2.$$

For given values of a *and* b *, the function* $f(x)$ *is continuous. The graph of the function*



is sketched in Figure .

1.3 Limit Exercises

1. Evaluate $\lim_{x \to 2}(4x^2 - 3x + 1)$
   Since this is a polynomial, two can simply be plugged in. This results in $4(4) - 2(3) + 1 = 16 - 6 + 1 = \mathbf{11}$

2. Evaluate $\lim_{x \to 5}(x^2)\ 5^2 = \mathbf{25}$

One-Sided Limits

Evaluate the following limits or state that the limit does not exist.

3. $\lim_{x \to 0^-} \dfrac{x^3+x^2}{x^3+2x^2}$
   Factor as $\dfrac{x^2\ x+1}{x^2\ x+2}$. In this form we can see that there is a removable discontinuity at x=0 and

that the limit is $\mathbf{\frac{1}{2}}$

4. $\lim_{x\to 7^-} |x^2+x|-x$
$|7^2+7|-7=\mathbf{49}$

5. $\lim_{x\to -1^-} \sqrt{1-x^2}$
$\sqrt{1-x^2}$ is defined if $x^2<1$, so the limit is $\sqrt{1-1^2}=\mathbf{0}$

6. $\lim_{x\to -1^+} \sqrt{1-x^2}$
$\sqrt{1-x^2}$ is not defined if $x^2>1$, so the limit does not exist.

Two-Sided Limits

Evaluate the following limits or state that the limit does not exist.

7. $\lim_{x\to -1} \frac{1}{x-1}$
$-\mathbf{\frac{1}{2}}$

8. $\lim_{x\to 4} \frac{1}{x-4}$
$\lim_{x\to 4^-} \frac{1}{x-4}=-\infty \lim_{x\to 4^+} \frac{1}{x-4}=+\infty$ The limit does not exist.

9. $\lim_{x\to 2} \frac{1}{x-2}$
$\lim_{x\to 2^-} \frac{1}{x-2}=-\infty \lim_{x\to 2^+} \frac{1}{x-2}=+\infty$.The limit does not exist.

10. $\lim_{x\to -3} \frac{x^2-9}{x+3}$
$=\lim_{x\to -3} \frac{(x+3)(x-3)}{x+3}=\lim_{x\to -3} x-3$
$=-3-3=\mathbf{-6}$

11. $\lim_{x\to 3} \frac{x^2-9}{x-3}$
$\lim_{x\to 3} \frac{(x-3)(x+3)}{x-3}=\lim_{x\to 3} x+3=3+3=\mathbf{6}$

12. $\lim_{x\to -1} \frac{x^2+2x+1}{x+1}$
$\lim_{x\to -1} \frac{(x+1)(x+1)}{x+1}$
$=\lim_{x\to -1} x+1=-1+1=\mathbf{0}$

13. $\lim_{x\to -1} \frac{x^3+1}{x+1}$
$\lim_{x\to -1} \frac{(x^2-x+1)(x+1)}{x+1}$
$=\lim_{x\to -1} x^2-x+1=(-1)^2-(-1)+1$
$=1+1+1=\mathbf{3}$

14. $\lim_{x\to 4} \frac{x^2+5x-36}{x^2-16}$
$\lim_{x\to 4} \frac{(x-4)(x+9)}{(x-4)(x+4)}=\lim_{x\to 4} \frac{x+9}{x+4}=\frac{4+9}{4+4}=\mathbf{\frac{13}{8}}$

15. $\lim_{x\to 25} \frac{x-25}{\sqrt{x}-5}$
$\lim_{x\to 25} \frac{(\sqrt{x}-5)(\sqrt{x}+5)}{\sqrt{x}-5}$
$=\lim_{x\to 25} \sqrt{x}+5)=\sqrt{25}+5)=5+5=\mathbf{10}$

16. $\lim_{x\to 0} \frac{|x|}{x}$
$\lim_{x\to 0^-} \frac{|x|}{x}$
$=\lim_{x\to 0^-} \frac{-x}{x}=\lim_{x\to 0^-} -1=-1$
$\lim_{x\to 0^+} \frac{|x|}{x}=\lim_{x\to 0^+} \frac{x}{x}=\lim_{x\to 0^+} 1=1.$ The limit does not exist.

17. $\lim_{x\to 2} \frac{1}{(x-2)^2}$
As x approaches 2, the denominator will be a very small positive number, so the whole fraction will be a very large positive number. Thus, the limit is ∞.

18. $\lim_{x\to 3} \frac{\sqrt{x^2+16}}{x-3}$
As x approaches 3, the numerator goes to 5 and the denominator goes to 0. Depending on whether you approach 3 from the left or the right, the denominator will be either a very small negative number, or a very small positive number. So the limit from the left is $-\infty$ and the limit from the right is $+\infty$. Thus, the limit does not exist.

19. $\lim_{x\to -2} \frac{3x^2-8x-3}{2x^2-18}$
$\frac{3(-2)^2-8(-2)-3}{2(-2)^2-18}$
$=\frac{3(4)+16-3}{2(4)-18}=\frac{12+16-3}{8-18}=\frac{25}{-10}=-\mathbf{\frac{5}{2}}$

20. $\lim_{x\to 2} \frac{x^2+2x+1}{x^2-2x+1}$
$\frac{2^2+2(2)+1}{2^2-2(2)+1}=\frac{4+4+1}{4-4+1}=\frac{9}{1}=\mathbf{9}$

21. $\lim_{x\to 3} \frac{x+3}{x^2-9}$
$\lim_{x\to 3} \frac{x+3}{(x+3)(x-3)}=\lim_{x\to 3} \frac{1}{x-3} \lim_{x\to 3^-} \frac{1}{x-3}=-\infty$
$\lim_{x\to 3^+} \frac{1}{x-3}=+\infty$. The limit does not exist.

22. $\lim_{x\to -1} \frac{x+1}{x^2+x}$
$\lim_{x\to -1} \frac{x+1}{x(x+1)}=\lim_{x\to -1} \frac{1}{x}=\frac{1}{-1}=\mathbf{-1}$

23. $\lim_{x\to 1} \frac{1}{x^2+1}$
$\frac{1}{1^2+1}=\frac{1}{1+1}=\mathbf{\frac{1}{2}}$

24. $\lim_{x\to 1} x^3+5x-\frac{1}{2-x}$
$1^3+5(1)-\frac{1}{2-1}=1+5-\frac{1}{1}=6-1=\mathbf{5}$

25. $\lim_{x\to 1} \frac{x^2-1}{x^2+2x-3}$
$\lim_{x\to 1} \frac{(x-1)(x+1)}{(x-1)(x+3)}=\lim_{x\to 1} \frac{x+1}{x+3}=\frac{1+1}{1+3}=\frac{2}{4}=\mathbf{\frac{1}{2}}$

26. $\lim_{x\to 1} \frac{5x}{x^2+2x-3}$
Notice that as x approaches 1, the numerator approaches 5 while the denominator approaches 0. However, if you approach from below, the denominator is negative, and if you approach from above, the denominator is positive. So the limits from the left and right will be $-\infty$ and $+\infty$ respectively. Thus, the limit does not exist.

Limits to Infinity

Evaluate the following limits or state that the limit does not exist.

27. $\lim_{x\to\infty} \frac{-x+\pi}{x^2+3x+2}$
This rational function is bottom-heavy, so the limit is **0**.

28. $\lim_{x\to-\infty} \frac{x^2+2x+1}{3x^2+1}$
This rational function has evenly matched powers of x in the numerator and denominator, so the limit will be the ratio of the coefficients, i.e. $\boldsymbol{\frac{1}{3}}$.

29. $\lim_{x\to-\infty} \frac{3x^2+x}{2x^2-15}$
Balanced powers in the numerator and denominator, so the limit is the ratio of the coefficients, i.e. $\boldsymbol{\frac{3}{2}}$.

30. $\lim_{x\to-\infty} 3x^2-2x+1$
This is a top-heavy rational function, where the exponent of the ratio of the leading terms is 2. Since it is even, the limit will be ∞.

31. $\lim_{x\to\infty} \frac{2x^2-32}{x^3-64}$
Bottom-heavy rational function, so the limit is **0**.

32. $\lim_{x\to\infty} 6$
This is a rational function, as can be seen by writing it in the form $\frac{6x^0}{1x^0}$. Since the powers of x in the numerator and denominator are evenly matched, the limit will be the ratio of the coefficients, i.e. **6** .

33. $\lim_{x\to\infty} \frac{3x^2+4x}{x^4+2}$
Bottom-heavy, so the limit is **0**.

34. $\lim_{x\to-\infty} \frac{2x+3x^2+1}{2x^2+3}$
Evenly matched highest powers of x in the numerator and denominator, so the limit will be the ratio of the corresponding coefficients, i.e. $\boldsymbol{\frac{3}{2}}$.

35. $\lim_{x\to-\infty} \frac{x^3-3x^2+1}{3x^2+x+5}$
Top-heavy rational function, where the exponent of the ratio of the leading terms is 1, so the limit is $-\infty$.

36. $\lim_{x\to\infty} \frac{x^2+2}{x^3-2}$
Bottom-heavy, so the limit is **0**.

Limits of Piecewise Functions

Evaluate the following limits or state that the limit does not exist.

37. Consider the function $f(x) = \begin{cases} (x-2)^2 & \text{if } x<2 \\ x-3 & \text{if } x\geq 2. \end{cases}$

   (a) $\lim_{x\to 2^-} f(x)$
   $(2-2)^2 = \mathbf{0}$

   (b) $\lim_{x\to 2^+} f(x)$
   $2-3 = \mathbf{-1}$

   (c) $\lim_{x\to 2} f(x)$

Since the limits from the left and right don't match, the limit does not exist.

38. Consider the function

$$g(x) = \begin{cases} -2x+1 & \text{if } x\leq 0 \\ x+1 & \text{if } 0<x<4 \\ x^2+2 & \text{if } x\geq 4. \end{cases}.$$

   (a) $\lim_{x\to 4^+} g(x)$
   $4^2+2 = 16+2 = \mathbf{18}$

   (b) $\lim_{x\to 4^-} g(x)$ $4+1 = \mathbf{5}$

   (c) $\lim_{x\to 0^+} g(x)$ $0+1 = \mathbf{1}$

   (d) $\lim_{x\to 0^-} g(x)$
   $-2(0)+1 = \mathbf{1}$

   (e) $\lim_{x\to 0} g(x)$ Since the left and right limits match, the overall limit is also **1** .

   (f) $\lim_{x\to 1} g(x)$

39. Consider the function $h(x) = \begin{cases} 2x-3 & \text{if } x<2 \\ 8 & \text{if } x=2 \\ -x+3 & \text{if } x>2. \end{cases}$

   (a) $\lim_{x\to 0} h(x)$
   $2(0)-3 = \mathbf{-3}$

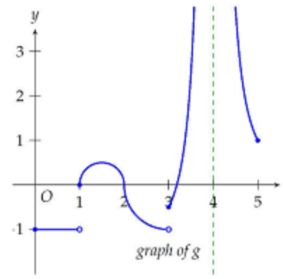
   (b) $\lim_{x\to 2^-} h(x)$
   $2(2)-3 = 4-3 = \mathbf{1}$

   (c) $\lim_{x\to 2^+} h(x)$
   $-(2)+3 = \mathbf{1}$

   (d) $\lim_{x\to 2} h(x)$

Since the limits from the right and left match, the overall limit is **1** . Note that in this case, the limit at 2 does not match the function value at 2, so the function is discontinuous at this point, hence the function is non differentiable at this point as well.

40. The graph of a function g is shown.



(a) At which points a in $(0, 1, 2, 3, 4, 5)$ is g continuous?

(b) At which points a in $(0, 1, 2, 3, 4, 5)$ is g continuous from the right?

(c) At which points a in $(0, 1, 2, 3, 4, 5)$ is g continuous from the left?